Portfolio Selection using Kernel Regression

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ABSTRACT

- We use kernel regression to improve the performance of indexes
- Utilizing recent price history can improve the returns
Portfolio Selection

- We have $d$ assets $S^1_t, \ldots, S^d_t$

- Return vector

$$R_t = \left( \frac{S^1_t - S^1_{t-1}}{S^1_{t-1}}, \ldots, \frac{S^d_t - S^d_{t-1}}{S^d_{t-1}} \right)$$

- Portfolio vector $b_{t-1} \in \mathbb{R}^d$

- Portfolio returns

$$b^T_{t-1} R_t = b^1_{t-1} R^1_t + \cdots + b^d_{t-1} R^d_t$$
Markowitz and Expected Utility

Markowitz: Find portfolio vector \( \mathbf{b} \) that maximizes

\[
E (b^T R_{t+1}) - \frac{\gamma}{2} \text{Var} (b^T R_{t+1})
\]

Expected utility: Find \( \mathbf{b} \) that maximizes

\[
Eu (b^T R_{t+1})
\]
Utility Functions

• CRRA utility functions

\[ u(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma}, & \text{if } \gamma > 1, \\ \log_e w, & \text{if } \gamma = 1. \end{cases} \]

• CARA utility functions

\[ u(w) = 1 - e^{-\alpha w}, \quad \alpha > 0. \]
Define the conditional expected utility

\[ f_b(x) = E (u(b^T R_{t+1}) \mid X_t = x) \]

Find the portfolio vector as

\[ b_t = \text{argmax}_b f_b(x_t) \]

The explanatory variables can be chosen as

\[ X_t = (R_t, \ldots, R_{t-k+1}) \in \mathbb{R}^{kd} \]
We have data for $t = 0, \ldots, t_0 - 1$

$$Y_t = u(b^T R_{t+1}), \quad X_t = (R_t, \ldots, R_{t-k+1})$$

We estimate the regression function by

$$\hat{f}_b(x) \approx E \left( u(b^T R_{t+1}) \mid X_t = x \right)$$

Define the portfolio vector as

$$\hat{b}_{t_0} = \arg\max_b \hat{f}_b(x_{t_0})$$


Kernel Regression

Kernel estimate is a weighted average

$$\hat{f}(x) = \sum_{t=0}^{t_0-1} p_t(x) Y_t$$

Weights are defined as

$$p_t(x) = \frac{K_h(x - X_t)}{\sum_{u=0}^{t_0-1} K_h(x - X_u)}$$

Scaled kernel

$$K : \mathbb{R}^d \rightarrow \mathbb{R}, \quad K_h(x) = K(x/h)/h^d$$
S&P 500, NASDAQ-100

- Data between 1985-01-01 and 2009-05-04
- S&P 500: annualized mean 8.8%, standard deviation 18.9%, Sharpe ratio 0.47
- NASDAQ-100: annualized mean 15.2%, standard deviation 28.8%, Sharpe ratio 0.53
S&P 500, Nasdaq-100

- Smoothing parameter $h = 0.5, 1, 2$
- Autoregression parameter $k = 5, 10$.
- Best Sharpe ratio 0.79 and the worst 0.64.

Figure 1: Wealth against time from October 1985 to May 2009. The market portfolio is shown as the black line.
Let us have $L$ parameter combinations that give $L$ sequences of portfolio weights

$$b_0^{(l)}, \ldots, b_{t_0}^{(l)}, \ l = 1, \ldots, L$$

The corresponding wealth processes

$$W_{t_0}^{(l)} = W_0 \prod_{t=0}^{t_0-1} (R_{t+1} + 1)^T b_t^{(l)}$$

The portfolio vector obtained by combining strategies is

$$b_t = \sum_{l=1}^L q_t^{(l)} b_t^{(l)}, \ t = 1, \ldots, t_0$$

where

$$q_t^{(l)} = \frac{W_{t-1}^{(l)}}{\sum_{l=1}^L W_{t-1}^{(l)}}$$
S&P 500, NASDAQ-100

- Long only, without bank account
- Annualized mean is 18.1%, standard deviation 24.3%, Sharpe ratio 0.74.
**Portfolio Weights**

* Simple strategies have mostly 0-1-weights

![Diagram](image-url)
S&P 500, Nasdaq-100

- Bank account included
- Smoothing parameter $h = 0.5, 1, 2$
  Autoregression parameter $k = 5, 10$.
- Best Sharpe ratio 0.98 and the worst 0.62.
S&P 500, Nasdaq-100

- Combining strategies
- Long only, with bank account
- Annualized mean is 18.1%, standard deviation 20.3%, Sharpe ratio 0.89
Portfolio Weights

Simple strategies have mostly 0-1-weights
SP 500, NASDAQ-100

- Shorting allowed
- Smoothing parameter $h = 0.7, 1, 1.5, 2$
- Autoregression parameter $k = 10$.
- Best Sharpe ratio 0.87 and the worst 0.74.
SP500, NASDAQ-100

- Combining strategies
- Shorting allowed
- Annualized mean is 20.4%, standard deviation 22.3%, Sharpe ratio 0.91.

![Graphs showing SP500 and NASDAQ-100 wealth over time. The portfolio is shown as a red line, and the market is shown as a black line.](image.png)
S&P 500, NASDAQ-100

- Combining strategies, shorting allowed
- Left and right tail plots
- red=S&P 500, blue=NASDAQ-100, black=portfolio

Figure 1: The distribution of returns that allows shorting. The period during 1 October 1985 until 4 May 2009.
Table 6: Quantiles of the SP500–NASDAQ-100 portfolio allowing shorting.

<table>
<thead>
<tr>
<th>Quantile</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.05</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.8%</td>
<td>-2.0%</td>
<td>-1.3%</td>
<td>0.06%</td>
<td>1.4%</td>
<td>2.1%</td>
<td>4.4%</td>
</tr>
</tbody>
</table>

Table 15: Empirical quantiles of the S&P 500 returns.

<table>
<thead>
<tr>
<th>Quantile</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.05</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2.6%</td>
<td>-1.4%</td>
<td>-1.0%</td>
<td>0.04%</td>
<td>1.0%</td>
<td>1.4%</td>
<td>2.6%</td>
</tr>
</tbody>
</table>

Table 18: Empirical quantiles of the NASDAQ-100 returns.
Biggest losses

<table>
<thead>
<tr>
<th>-10.9%</th>
<th>-9.2%</th>
<th>-8.3%</th>
<th>-7.7%</th>
<th>-7.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6.8%</td>
<td>-6.7%</td>
<td>-6.7%</td>
<td>-6.5%</td>
<td>-6.5%</td>
</tr>
<tr>
<td>2009-03-23</td>
<td>2008-11-20</td>
<td>2001-03-20</td>
<td>2002-07-29</td>
<td>2000-12-08</td>
</tr>
</tbody>
</table>

Table 4: The 10 biggest daily losses of the SP500–NASDAQ-100 portfolio allowing shorting and their dates.

<table>
<thead>
<tr>
<th>-20.5%</th>
<th>-9.0%</th>
<th>-8.9%</th>
<th>-8.8%</th>
<th>-8.3%</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7.6%</td>
<td>-6.9%</td>
<td>-6.8%</td>
<td>-6.8%</td>
<td>-6.7%</td>
</tr>
</tbody>
</table>

Table 13: The 10 biggest daily losses of S&P 500 and their dates.

<table>
<thead>
<tr>
<th>-15.1</th>
<th>-11.7</th>
<th>-10.5</th>
<th>-9.9</th>
<th>-9.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.1</td>
<td>-8.8</td>
<td>-8.3</td>
<td>-8.0</td>
<td>-7.9</td>
</tr>
<tr>
<td>2001-01-02</td>
<td>2008-10-15</td>
<td>2001-09-17</td>
<td>2008-12-01</td>
<td>2000-12-20</td>
</tr>
</tbody>
</table>

Table 16: The 10 biggest daily losses of NASDAQ-100 and their dates.
### Biggest Daily Gains

<p>| | | | | | |</p>
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>20.5%</td>
<td>18.8%</td>
<td>12.6%</td>
<td>10.6%</td>
<td>9.8%</td>
<td></td>
</tr>
<tr>
<td>9.3%</td>
<td>9.10%</td>
<td>9.09%</td>
<td>8.8%</td>
<td>8.4%</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: The 10 biggest daily gains of the SP500–NASDAQ-100 portfolio allowing shorting and their dates.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>11.6</td>
<td>10.8</td>
<td>9.1</td>
<td>7.1</td>
<td>6.9</td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td>6.4</td>
<td>6.3</td>
<td>5.7</td>
<td>5.4</td>
<td></td>
</tr>
</tbody>
</table>

Table 14: The 10 biggest daily gains of S&P 500 and their dates.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>18.8</td>
<td>12.6</td>
<td>11.7</td>
<td>10.9</td>
<td>10.8</td>
<td></td>
</tr>
<tr>
<td>2001-01-03</td>
<td>2008-10-13</td>
<td>2000-12-05</td>
<td>2008-10-28</td>
<td>2001-04-05</td>
<td></td>
</tr>
<tr>
<td>10.6</td>
<td>10.3</td>
<td>10.1</td>
<td>10.0</td>
<td>9.5</td>
<td></td>
</tr>
</tbody>
</table>

Table 17: The 10 biggest daily gains of NASDAQ-100 and their dates.
S&P 500, NASDAQ-100

- Combining strategies, shorting allowed
- Left and right tail plots for different risk aversion parameters gamma
- black: $\gamma = 1$, red: $\gamma = 25$, blue: $\gamma = 100$
- Sharpe ratios: 0.91, 0.87, and 0.61
Figure 15: Expected utility curves for the portfolio with $\gamma = 1$ (black), for the portfolio with $\gamma = 10$ (red), for the S&P 500 (green), and for the NASDAQ-100 (blue). The portfolio is the S&P 500–NASDAQ-100 portfolio with shorting allowed. The period extends from 1 October 1985 to 4 May 2009.
**Euro, UK Sterling**

- Shorting allowed, combining strategies
- 1999-04-01 to 2009-08-07
- Mean ret 4.8%, sd 7.4%, Sharpe ratio 0.65
Shorting allowed, combining strategies
Ret 11.4%, sd 9.8%, Sharpe ratio 1.16
Summary

- We can improve market portfolios using kernel regression
- The dependence between indexes can be utilized